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On tachyons in generic orbifolds of $\mathbb{C}^r$ and gauged linear sigma models

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Abstract: We study some aspects of Gauged Linear Sigma Models corresponding to orbifold singularities of the form $\mathbb{C}^r/\Gamma$, for $r = 2, 3$ and $\Gamma = \mathbb{Z}_n$ and $\mathbb{Z}_n \times \mathbb{Z}_m$. These orbifolds might be tachyonic in general. We compute expressions for the multi parameter sigma model Lagrangians for these orbifolds, in terms of their toric geometry data. Using this, we analyze some aspects of the phases of generic orbifolds of $\mathbb{C}^r$.

Keywords: Tachyon Condensation, Conformal Field Models in String Theory, D-branes.

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Contents

1. Introduction

Tachyon dynamics have been studied extensively in the last few years, following the pioneering work of Sen [1], and is important for the general understanding of the role of time in string theory. Whereas open string tachyon condensation processes lead to the decay of D-branes, the analogous condensation of closed string tachyons (that breaks space-time supersymmetry) result in a decay of spacetime itself. The general problem of studying the dynamics of tachyons in closed string theories is quite difficult, since there might be delocalised tachyons in the theory, but a class of theories where the problem is more tractable are theories with localised closed string tachyons. This was the issue addressed by Adams, Polchinski and Silverstein (APS) [2].

The approach of APS was to study the brane probe picture of string theories with localised tachyons, which arise as twisted sector states in the corresponding closed string theory. Clearly, the problem involves two distinct scales. In the substringy regime, where α' corrections are small, D-branes probing the orbifold in question provide a good description to the singularity structure via the gauge theory living on the brane. Far from the substringy regime, the brane probe picture is less useful and one has to resort to a supergravity analysis. In any case, in the open string picture, one finds that by exciting marginal or tachyonic deformations in the theory, one can drive the original orbifold to one of lower rank, and tachyonic deformations of the latter takes the system to a final stable (supersymmetric) configuration[1]

[1] This is strictly true only for $\mathbb{C}/\mathbb{Z}_n$ and $\mathbb{C}^2/\mathbb{Z}_n$ orbifolds. For $\mathbb{C}^3/\mathbb{Z}_n$ orbifolds, in the absence of a canonical resolution, there might be terminal singularities, i.e the end point of tachyon condensation need not result in a singularity that can be resolved solely by marginal deformations.

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An useful alternative way to study the decay of localised closed string tachyons is to focus on the $N = (2, 2)$ CFT of the worldsheet. By using an appropriate Gauged Linear Sigma Model (referred to as the GLSM in the sequel), one can effectively track down the decays of closed string tachyons [3 – 5] (for reviews, see [6, 7]). For orbifolds of the form $\mathbb{C}^3/\mathbb{Z}_n$, this analysis was carried out in [8, 9], and it was shown in [8] that generically, while the decay products of Type II string theories with tachyons in the spectrum can be deformed to flat geometries using only marginal deformations of the resulting theory at the endpoint of the tachyon decay, Type 0 theories might have terminal singularities as the endpoint of the tachyon decay process. Further, in [10], the GLSM was used to effectively study the phases of the theory, (for two and three parameter GLSMs), and a rich phase structure of the GLSM was found. In [9], a slightly different approach to the problem of tachyon condensation in the non-supersymmetric $\mathbb{C}^2/\mathbb{Z}_n$ and $\mathbb{C}^3/\mathbb{Z}_n$ was followed, and the classical sigma model metrics for the GLSMs were calculated, which effectively described the condensation of localised closed string tachyons in the same. Here, a single parameter GLSM (with a single $\mathrm{U}(1)$ gauge group) was considered, corresponding to the most relevant tachyon (i.e the one with the highest (negative) mass squared). Condensation of the most relevant tachyon of course leaves the possibility of then exciting other tachyonic modes (which remain tachyonic at the end of the decay).

In general, the GLSM describing the geometry of the instability due to closed string tachyons will be charged under multiple $\mathrm{U}(1)$ gauge groups, and it is important to have an full understanding of the same, in order to understand the phase structures of these, in lines with [10]. An advantage of doing this is that one can then study the phase structure of arbitrary charge GLSMs, without resorting to a case by case analysis. Further, the non-linear sigma model metrics for these unstable geometries can possibly be understood via a generic toric description. It is this issue that we set out to address in this paper. In particular, we will show how to construct the bosonic Lagrangian for generic multi parameter GLSMs with arbitrary charges, entirely in terms of the toric data of the target space singular geometry that the GLSM describes. This will help us to read off the phase structures of the corresponding theories and their classical metrics. On the other hand, D-brane dynamics in generic orbifolds of $\mathbb{C}^r$ can be understood in terms of open string GLSM boundary conditions, as shown in refs. [11, 12]. As we will see later, our results here can be used to extend the analysis of these papers to arbitrary charged GLSMs, and might be important in the study of D-brane dynamics in such models.

The paper is organised as follows. In section 2, we review the toric construction of singularities of $\mathbb{C}^2$ and $\mathbb{C}^3$, in order to set the notation and conventions used in the rest of the paper. In section 3, we first construct the bosonic Lagrangian for one and two parameter GLSMs, and then move on to construct the Lagrangian for a generic multi-parameter GLSM with arbitrary charges. In section 4, we apply the construction of section 3, to study some aspects of the phases of generic orbifolds of $\mathbb{C}^3$. Section 5 contains our conclusions and discussions.

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2. Closed string tachyons on $\mathbb{C}^r/\Gamma$

In this section, we state some of the relevant results for closed string tachyon condensation of orbifolds of the form $\mathbb{C}^r/\Gamma$, for $\Gamma = \mathbb{Z}_n$ or $\mathbb{Z}_m \times \mathbb{Z}_n$. This section is primarily review material, in order to set the notations and conventions used in the remainder of the paper.

String theory on the space-time non-supersymmetric orbifold $\mathbb{R}^{7,1} \times \mathbb{C}/\mathbb{Z}_n$ with the quotienting group acting on a complexified direction Z with the action

$$Z \to \omega Z; \qquad \omega = e^{\frac{2\pi i}{n}} \tag{2.1}$$

was studied by Adams, Polchinski and Silverstein (APS) [2]. The closed string conformal field theory is tachyonic in all its twisted sectors, and localisation of the tachyons demands that n be odd in Type II theories. The world volume gauge theory of a D-p brane probing this singularity (the brane being located at the orbifold fixed point) can be constructed a'la Douglas and Moore [13], and the orbifolding action will retain only those fields invariant under its action, thus resulting in a quiver gauge theory on the D-brane world volume. APS showed that by giving vevs to certain scalar fields in the resulting gauge theory, some of the other fields become massive (as can be determined from the scalar potential of the gauge theory), and the remaining massless fields describe the quiver gauge theory of a lower rank orbifold. Fermionic fields in the gauge theory are analysed similarly, using the Yukawa coupling terms. The probe brane analysis is valid at substringy regimes, and when gravity effects become large, one has to resort to supergravity techniques. The two yield a consistent description of the decay of the conical singularity into flat space-time, as demonstrated by APS (see also [14, 15]).

The case of complex two fold orbifolds yields a richer structure (see, e.g. [2, 4, 16]). In this case, we consider string theory in the background $\mathbb{R}^{5,1} \times \mathbb{C}^2/\mathbb{Z}_n$, where $\mathbb{R}^{5,1}$ is flat six dimensional Minkowski space-time, and the remaining four directions are complexified, with the orbifolding group acting as

$$\left(Z^1, Z^2\right) \to \left(\omega Z^1, \omega^k Z^2\right), \qquad \omega = e^{\frac{2\pi i}{n}} \tag{2.2}$$

this orbifold action (denoted by $\mathbb{C}^2/\mathbb{Z}_{n(k)}$) breaks space-time supersymmetry, whenever $1 + k \neq 0 \mod n$. Again the open string analysis in the substringy regime can be carried out in the same way as in the $\mathbb{C}/\mathbb{Z}_n$ case. The APS procedure here is to give vevs to some of the scalar fields by first turning on marginal deformations in the theory that breaks a part of the original orbifolding group, and takes the theory to a locally supersymmetric, lower rank orbifold. Deformations of the latter, which are tachyonic, then drives the system to a supersymmetric background. The supergravity analysis in this case is more complicated than in the case of the $\mathbb{C}/\mathbb{Z}_n$ orbifold, and one has to possibly resort to numerical techniques in order to study the same. A similar analysis can be carried out for $\mathbb{C}^3$ orbifolds as well.

In [17], a brane probe analysis was carried out for the non-supersymmetric orbifolds of APS, following the procedure of [18, 19]. It was found that the procedure of [19] yields the correct toric geometry (see refs [20], for comprehensive introductions to toric varieties) from the gauge theory of D-branes probing non-supersymmetric orbifolds, but interestingly with certain marginal directions being turned on. This complemented the work of [3 − 5], where an equivalent picture of the APS decay process was given in terms of the $(2, 2)$ world sheet superconformal field theory (SCFT) of closed strings on non-supersymmetric orbifolds, which is closely related to the toric description of such orbifolds, and in turn relates directly to the gauged linear sigma model of Witten [21]. The open string description in lines with [19] becomes technically complicated for generic orbifolds, and in this paper we will primarily focus on the equivalent closed string description.

The closed string SCFT description of non-supersymmetric orbifolds essentially consists of turning on the twisted sectors of the theory that correspond to relevant (or marginal) deformations, and studying the decay of the orbifolds under these [4]. In the NSR formalism for closed string theory on generic orbifolds, the tachyonic deformations correspond to the orbifold twisted chiral operators with the (total) R-charge less than unity, while marginal deformations will have the total R-charge adding to one. Turning on these deformations, one can study the RG flow of the resulting theory, and these nicely describe the decay of non-supersymmetric orbifolds [4]. This is in turn closely related to the toric geometry of non-supersymmetric orbifolds. Indeed, It was shown in [5] that the generators of the chiral ring for these orbifolds are in one to one correspondence with the minimal resolution curves of the singularities of $\mathbb{C}^2/\mathbb{Z}_n$, and the R-charges of these are closely related to the integers appearing in the Hirzebruch-Jung continued fraction that provides the intersection numbers of the $\mathbb{CP}^1$s involved in the resolution of the singularity.

Specifically, the toric geometry of the closed string SCFT probing generic orbifolds of $\mathbb{C}^2$ or $\mathbb{C}^3$ can be studied by considering the orbifold twisted sectors, that are relevant (or marginal). The toric data for the resolution of these orbifolds can be obtained most simply by adding certain fractional points corresponding to the orbifold action (in a $\mathbb{Z}^{\oplus r}$ lattice for an orbifold of $\mathbb{C}^r$) and then restoring integrality in the lattice (see, e.g. [22 − 24]). Let us see if we can substantiate this. Consider the orbifold $\mathbb{C}^2/\mathbb{Z}_{5(2)}$. In this case, there are four twisted sectors, and the relevant deformations correspond to those with R-charges [4]

$$\left(\frac{1}{5},\frac{2}{5}\right), \quad \left(\frac{3}{5},\frac{1}{5}\right), \tag{2.3}$$

and these are the generators of the chiral ring of the orbifold SCFT. The toric data for this orbifold is obtained by considering the integral $\mathbb{Z}^{\oplus 2}$ lattice generated by the vectors $\vec{e}_1, \vec{e}_2 = (1,0), (0,1)$, but now with the fractional points $\left(\frac{1}{5},\frac{2}{5}\right), \left(\frac{3}{5},\frac{1}{5}\right)$, which we call $\vec{e}_3, \vec{e}_4$ respectively. In order to restore integrality in the augmented lattice, we now express the vectors in terms of $\vec{e}_3$, $\vec{e}_2$, which we now label as $(1,0), (0,1)$ respectively. Doing this, we get the toric data as

$$\mathcal{T} = \begin{pmatrix} 1 & 0 & -1 & -3 \\ 0 & 1 & 3 & 5 \end{pmatrix} \tag{2.4}$$

This is easily seen to be the toric data for the orbifold $\mathbb{C}^2/\mathbb{Z}_{5(2)}$, corresponding to the continued fraction $\frac{5}{2}$.[2]

[2] In general for more complicated orbifold theories, there might be more than one way to restore integrality in a lattice. These are essentially row-equivalent, but as a curiosity, we note that the number of these possible operations might be related to the multiplicities of the GLSM fields found via the open string picture in [17].

A similar analysis can be done for orbifolds of the form $\mathbb{C}^3/\mathbb{Z}_m \times \mathbb{Z}_n$, both for the space-time supersymmetric and non-supersymmetric cases. In this case, there will be two generators of the orbifold action, which we will choose to be

$$\begin{aligned} g_1 &: \left(Z^1, Z^2, Z^3\right) \to \left(\omega_1 Z^1, \omega_1^p Z^2, Z^3\right) \\ g_2 &: \left(Z^1, Z^2, Z^3\right) \to \left(\omega_2 Z^1, Z^2, \omega_2^q Z^3\right) \end{aligned} \tag{2.5}$$

where $\omega_1 = e^{\frac{2\pi i}{m}}, \omega_2 = e^{\frac{2\pi i}{n}}$, and p and q are integers.[3] Again, as a concrete example, let us consider the space-time supersymmetric orbifold $\mathbb{C}^3/\mathbb{Z}_3 \times \mathbb{Z}_3$, [25, 26] with the orbifolding action being

$$\begin{aligned} g_1 &: \left(Z^1, Z^2, Z^3\right) \to \left(\omega_1 Z^1, \omega_1^2 Z^2, Z^3\right) \\ g_2 &: \left(Z^1, Z^2, Z^3\right) \to \left(\omega_2 Z^1, Z^2, \omega_2^2 Z^3\right) \end{aligned} \tag{2.6}$$

where now $\omega = e^{\frac{2\pi i}{3}}$. In the brane probe picture, one considers the regular representations of the group $\mathbb{Z}_3 \times \mathbb{Z}_3$ which acts non trivially on the space-time coordinates as well as the Chan-Paton factors. Equivalently, in the closed string description, we consider the $\mathbb{Z}^{\oplus 3}$ lattice, generated by $\vec{e}_1 = (1,0,0)$, $\vec{e}_2 = (0,1,0)$, $\vec{e}_3 = (0,0,1)$, but now including the following seven fractional points $\vec{e}_4, \ldots, \vec{e}_{10}$ which correspond to the seven surviving (marginal) sectors in the theory, in our lattice:

$$\begin{aligned} &\left(\frac{1}{3}, \frac{2}{3}, 0\right), \ \left(\frac{2}{3}, \frac{1}{3}, 0\right), \ \left(\frac{1}{3}, 0, \frac{2}{3}\right), \ \left(\frac{2}{3}, 0, \frac{1}{3}\right), \\ &\left(0, \frac{1}{3}, \frac{2}{3}\right), \ \left(0, \frac{2}{3}, \frac{1}{3}\right), \ \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right) \end{aligned} \tag{2.7}$$

where in the second line, we have included the fractional points corresponding to the action by $g_1.g_2$ as well. In this case there are various possibilities of restoring integrality in our lattice. All these are of course related by $SL\left(3, \mathbb{R}\right)$ transformations. In particular, if we rewrite the vectors in terms of $\vec{e}_8$, $\vec{e}_6$, $\vec{e}_3$, which we now label by $(1,0,0)$, $(0,1,0)$, $(0,0,1)$ respectively, we obtain the toric data

$$\mathcal{T} = \begin{pmatrix} 1 & 0 & 0 & 2 & -1 & 2 & 3 & 1 & -2 & 4 \\ 0 & 1 & 0 & 2 & 0 & 1 & 2 & 1 & 0 & 3 \\ 0 & 0 & 1 & -3 & 2 & -2 & -4 & -1 & 3 & -6 \end{pmatrix} \tag{2.8}$$

After a few row operations, this can be recognised as the toric data for the resolution of the orbifold $\mathbb{C}^3/\mathbb{Z}_3 \times \mathbb{Z}_3$.

Similar analyses can be carried out for product orbifolds that have tachyons in some of the twisted sectors. Consider, for example, the non-cyclic orbifold $\mathbb{C}^3/\mathbb{Z}_5 \times \mathbb{Z}_5$, with the orbifold action now being given by

$$\begin{aligned} g_1 &: \left(Z^1, Z^2, Z^3\right) \to \left(\omega Z^1, \omega^2 Z^2, Z^3\right) \\ g_2 &: \left(Z^1, Z^2, Z^3\right) \to \left(Z^1, \omega Z^2, \omega^2 Z^3\right) \end{aligned} \tag{2.9}$$

[3] We will assume p and q to be positive in what follows.

where now $\omega = e^{\frac{2\pi i}{5}}$. The orbifolding action will now introduce tachyons in the closed string spectrum, corresponding to twisted sectors that are relevant. In this case, the toric data can be found from the set of points (apart from the generators of the $\mathbb{Z}^{\oplus 3}$ lattice:

$$\left(\frac{1}{5},\frac{2}{5},0\right),\ \left(\frac{3}{5},\frac{1}{5},0\right),\ \left(0,\frac{1}{5},\frac{2}{5}\right),\left(0,\frac{3}{5},\frac{1}{5}\right),\ \left(\frac{1}{5},0,\frac{1}{5}\right) \tag{2.10}$$

Where all the twisted sectors are tachyonic. Again, we can restore integrality in the augmented lattice in the same way as before, and the toric data corresponding to this orbifold is given by

$$\mathcal{T} = \begin{pmatrix} 1 & 0 & 0 & -1 & -1 & -2 & 3 & -3 \\ 0 & 1 & 0 & 1 & 2 & 1 & -1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 3 & 0 & 5 \end{pmatrix} \tag{2.11}$$

A similar analysis can be carried out for four fold orbifolds as well.

Once we obtain the toric data for a given orbifold, it is easy to construct the charges of the GLSM fields that describes (in a classical limit) the orbifold singularity. In particular, the GLSM charge matrix is given by the kernel of the toric data. Consider, for example, the simplest non-cyclic orbifold, $\mathbb{C}^3/\mathbb{Z}_2 \times \mathbb{Z}_2$, whose toric data is given by [24]

$$\mathcal{T} = \begin{pmatrix} 1 & 0 & 0 & 0 & 2 & 1 \\ 0 & 1 & 0 & -1 & 1 & 1 \\ 0 & 0 & 1 & 2 & -2 & -1 \end{pmatrix} \tag{2.12}$$

The GLSM charge matrix, which is the dual cone to the toric data, is obtained as the kernel of $\mathcal{T}$ and is given by

$$Q = \begin{pmatrix} 1 & 0 & 0 & -1 & -1 & 1 \\ 0 & 1 & 0 & -1 & 1 & -1 \\ 0 & 0 & 1 & 1 & -1 & -1 \end{pmatrix} \tag{2.13}$$

Here we have three marginal deformations in the world sheet SCFT.[4] We can now look at the partial resolutions of this singularity [28], by resolving certain points in the toric diagram. In this example, if we consider the toric data with two of the marginal sectors turned on, we obtain the GLSM charge matrix

$$Q = \begin{pmatrix} 1 & 1 & 0 & -2 & 0 \\ 0 & 1 & 1 & 0 & -2 \end{pmatrix} \tag{2.14}$$

which clearly specifies the action of the orbifolding group on $\mathbb{C}^3$.

The same analysis can be carried over in the case of non-supersymmetric orbifolds as well. The GLSM charge matrix can be written down in a similar fashion as above, and using this, we can carry out an analysis of the phases of the orbifold theory. Note that in general, we will have GLSMs charged under multiple $\mathrm{U}(1)$ gauge groups. For simplicity, we can, however, choose turn on a subset of the deformations (relevant or marginal). Consider

[4] The charge matrix of the GLSM can be written in various bases. We will choose the basis that makes the resolution of the singularity clearly visible.

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e.g. the non-supersymmetric orbifold $\mathbb{C}^3/\mathbb{Z}_5 \times \mathbb{Z}_5$ with the orbifolding action, the twisted sector R-charges and the toric data given by (2.9), (2.10) and (2.11) respectively. Turning on two of the twisted sectors with R-charges $\left(\frac{1}{5}, \frac{2}{5}, 0\right)$ and $\left(0, \frac{1}{5}, \frac{2}{5}\right)$, the GLSM charge matrix is given by

$$Q = \begin{pmatrix} 1 & 2 & 0 & -5 & 0 \\ 0 & 1 & 2 & 0 & -5 \end{pmatrix} \tag{2.15}$$

In the next section, we will elaborate on this, and study in details the structure of the GLSM with arbitrary charge matrices. This will enable us to write down the classical metrics seen by the GLSM in certain limits, and will help us to analyse the phases of the models.

Before we end this section, a few comments on GSO projections of the string theories under consideration is in order. In [8], it was shown that in order for the string theory to admit a Type II GSO projection in the orbifold $\mathbb{C}^3/\mathbb{Z}_{n(k_1,k_2,k_3)}$, we should have $\sum_{i=1}^{3} k_i =$ even. The GSO projection acts nontrivially on the twisted sectors and the necessary condition that these be preserved under a Type II GSO projection as well is that the integer part of the R-charges of these twisted sectors should be even. In the GLSM, this amounts to demanding that the sum of the U(1) charges for each gauge group be even, in order to admit a Type II GSO projection [27]. In the case of type 0 string theories, on the other hand, the presence of the bulk tachyon might complicate the dynamics of the decay. However, we assume in these cases that the delocalised bulk tachyon is tuned such that it does not affect the dynamics of the RG flow, which is driven solely by the twisted sector tachyons. Since our analysis of the next section is completely general, we will not mention this point explicitly in future, and the GSO projection will be understood from the context.

3. Sigma model metrics for multi U(1) GLSMs

In this section, we study in details the multi parameter GLSMs, generalising the results of [9, 7]. This will be useful for us when we consider the phases of generic orbifolds of the form $\mathbb{C}^3/\mathbb{Z}_n \times \mathbb{Z}_m$. We will first start by reviewing briefly the relevant details of Wittens's GLSM [21]. We then construct the bosonic GLSM Lagrangian for arbitrary charges, and proceed to evaluate the sigma model metrics for the same.

3.1 Single parameter GLSMs

Witten's Gauged Linear Sigma model [21] provides an effective tool for studying closed string tachyon condensation on orbifolds [3]. The action for the GLSM, with an Abelian gauge group $\mathrm{U}(1)^s$ is

$$S = \int d^2z d^4\theta \sum_i \bar{\Phi}_i \Phi_i - \sum_a \frac{1}{4e_a^2} \int d^2z d^4\theta \bar{\Sigma}_a \Sigma_a + \mathrm{Re}\left[it \int d^2z d^2\tilde{\theta}\Sigma \right] \tag{3.1}$$

where the Φ_i are chiral superfields, Σ_a are twisted chiral superfields ($a = 1 \cdots s$), $t = ir + \frac{\theta}{2\pi}$ is a complexified parameter involving the Fayet-Iliopoulos parameter r and the two dimensional θ angle, and we are considering a theory without any superpotential.

The $e^2 \to \infty$ limit of the GLSM is the non-linear sigma model (NLSM) limit, and in this limit, the gauge fields appearing in (3.1) appear as Lagrange multipliers. It is then possible to solve the D-term constraint in the classical limit $|r| \to \infty$, to read off the sigma model metric corresponding to the GLSM [7, 9]. It will be enough for our purpose to focus on the bosonic part of the action in (3.1), given by

$$S = -\int d^2 z D_\mu \bar{\phi}_i D^\mu \phi_i \tag{3.2}$$

and study this action, using the D-term constraints,

$$\sum_a Q_i^a \bar{\phi}_i \phi_i = 0 \tag{3.3}$$

where ϕ_i are the bosonic components of Φ_i and Q_i^a denote the charge of the ϕ_i with respect to the a th U(1).

Orbifolds of the type $\mathbb{C}^r/\Gamma$, with $r = 1, 2, 3$ can be effectively described by a single parameter GLSM, with the number of gauge groups being dictated by the nature of the singularity. In this subsection, we will consider the single parameter GLSM, which describes closed string tachyon condensation in $\mathbb{C}/\mathbb{Z}_n$ (with n being odd for localised tachyons [2]), and can also be used to describe condensation of a single tachyon in orbifolds of $\mathbb{C}^2$ and $\mathbb{C}^3$. Let us start by reviewing the process of closed string tachyon condensation in $\mathbb{C}/\mathbb{Z}_n$.

In this case, the GLSM consists of two fields, charged under a single U(1) gauge group, with charges $(1, -n)$, and satisfies the D-term constraint given by

$$|\phi_1|^2 - n|\phi_2|^2 + r = 0 \tag{3.4}$$

where r is the Fayet-Iliopoulos parameter of the theory. The D-term constraint is solved by

$$\phi_1 = \rho_1 e^{i\theta_1} \qquad \phi_2 = \rho_2 e^{i\theta_2} = \sqrt{\frac{\rho_1^2 + r}{n}} e^{i\theta_2} \tag{3.5}$$

The bosonic Lagrangian (3.2) can be used to solve classically for the gauge field, with the solution being, in this case,

$$V_\mu = \frac{\sum_i Q_i \left(\bar{\phi}_i \partial_\mu \phi_i - \phi_i \partial_\mu \bar{\phi}_i\right)}{2i \sum Q_i^2 |\phi_i|^2} \tag{3.6}$$

Upon substituting (3.5) in (3.6), and putting it back in the action (3.2), we obtain the Lagrangian

$$L = (\partial_\mu \rho_1)^2 + (\partial_\mu \rho_2)^2 + \frac{\rho_1^2 \rho_2^2 \left(Q_1 d\theta_2 - Q_2 d\theta_1\right)^2}{\left(Q_1^2 \rho_1^2 + Q_2^2 \rho_2^2\right)} \tag{3.7}$$

In the limit when $r \to \infty$, substituting for ρ_1 and ρ_n from (3.5) in (3.7, we recover the sigma model metric of the cone, $\mathbb{C}/\mathbb{Z}_n$,

$$ds^2 = d\rho_1^2 + \frac{\rho_1^2}{n^2} d\theta^2 \tag{3.8}$$

where $\theta = (nd\theta_2 - d\theta_1)$ is the gauge invariant angle. This geometry can also be seen from the D-term equation (3.4). For large r, ϕ_n acquires a large vev, and the gauge group $\mathrm{U}(1)$ is broken to $\mathbb{Z}_n$. A similar calculation [7] with $r \to -\infty$ shows that the space is now flat (after tachyon condensation), in agreement with the predictions of [2].

In the same way, single parameter GLSMs with more fields can be considered. It is easy to carry out the analysis of the sigma model metric in exactly the same way above [9] and we will write the result for the Lagrangian for a GLSM with m fields $\phi_i, i = 1, \dots m$ with $\mathrm{U}(1)$ charges $Q_i, \quad i = 1, \dots m$:

$$L = (\partial_\mu \rho_1)^2 + (\partial_\mu \rho_2)^2 + \cdots + (\partial_\mu \rho_m)^2 + \frac{\sum_{i<j} \rho_i^2 \rho_j^2 \left(Q_i d\theta_j - Q_j d\theta_i\right)^2}{\sum_i Q_i^2 \rho_i^2} \tag{3.9}$$

The above formula gives the single parameter GLSM Lagrangian for the singularity $\mathbb{C}^{m-1}/\mathbb{Z}_n$ and is the main result of this subsection.

The sigma model metrics corresponding to the various limits of the Fayet-Iliopoulos parameter can again be obtained by solving the D-term constraints. As an example, the supersymmetric orbifold $\mathbb{C}^3/\mathbb{Z}_{3(-1)}$, which is described by

$$Q_i = (1, 1, 1, -3)\,, \qquad |\phi_1|^2 + |\phi_2|^2 + |\phi_3|^2 - 3|\phi_4|^2 + r = 0 \tag{3.10}$$

can be seen, in the limit $r \to \infty$ to have the sigma model metric

$$ds^2 = d\rho_1^2 + d\rho_2^2 + d\rho_3^2 + \frac{\rho_1^2}{9} d\tilde{\theta}_1^2 + \frac{\rho_2^2}{9} d\tilde{\theta}_2^2 + \frac{\rho_3^2}{9} d\tilde{\theta}_3^2 \tag{3.11}$$

where $\tilde{\theta}_1 = (3\theta_1 + \theta_4)$, $\tilde{\theta}_2 = (3\theta_2 + \theta_4)$, $\tilde{\theta}_3 = (3\theta_3 + \theta_4)$ are the gauge invariant combinations of the original phases appearing in the solutions for the ϕ_is. The sigma model metric in the limit $r \to -\infty$ can also be read off from this Lagrangian, and gives (three copies of) the flat space metric. This corresponds to "blowing up" the singularity using marginal operators in the CFT, as in [4]. Away from the classical limits (i.e when r is small), the sigma model metrics receive quantum corrections [7]. For our purposes, it will be enough to consider only the $|r| \to \infty$ limits in the above Lagrangian.

For tachyonic orbifolds, the general procedure outlined above can be used to study the decays of the singular theory, under tachyon condensation. For example, the tachyonic orbifold $\mathbb{C}^2/\mathbb{Z}_{n(k)}, 1 + k \neq 0 \mathrm{mod} n$ can be modeled by the single parameter GLSM with three fields, with the $\mathrm{U}(1)$ charges being $Q_i = (1, k, -n)$. In this case, for the limit $r \to \infty$, the D-term constraint

$$|\phi_1|^2 + k|\phi_2|^2 - n|\phi_3|^2 + r = 0 \tag{3.12}$$

is solved by setting

$$\phi_1 = \rho_1 e^{i\theta_1}, \phi_2 = \rho_2 e^{i\theta_2}, \phi_3 = \rho_3 e^{i\theta_3} = \sqrt{\frac{r + \rho_1^2 + k\rho_2^2}{n}} \tag{3.13}$$

In the limit when $r \to \infty$, substituting (3.13) in (3.9), we recover the metric [9]

$$ds^2 = d\rho_1^2 + d\rho_2^2 + \frac{\rho_1^2}{n^2} d\tilde{\theta}_1^2 + \frac{\rho_2^2}{(n/k)^2} d\tilde{\theta}_2^2 \tag{3.14}$$

where the gauge invariant angles are now $\tilde{\theta}_1 = (\theta_3 + n\theta_1), \tilde{\theta}_2 = \left(\theta_3 + \frac{n}{k}\theta_1\right)$. The geometry is most easily visualised by making the gauge choice $\theta_n = 0$ which fixes the metric to be

$$ds^2 = d\rho_1^2 + d\rho_2^2 + \rho_1^2 d\theta_1^2 + \rho_2^2 d\theta_2^2 \tag{3.15}$$

but with the simultaneous identifications

$$\theta_1 \simeq \theta_1 + \frac{2\pi}{n}, \quad \theta_2 \simeq \theta_2 + \frac{2\pi k}{n} \tag{3.16}$$

The $r \to -\infty$ limit can be worked out in an entirely analogous way. In this case the D-term constraint (3.12) shows that ϕ_1 and ϕ_2 cannot be simultaneously zero, and choosing ϕ_k to be very large, the D-term constraint can be solved by setting

$$\phi_1 = \rho_1 e^{i\theta_1}, \phi_2 = \sqrt{\frac{\rho_3^2 - \rho_1^2 - r}{k}}, \phi_3 = \rho_3 e^{i\theta_3} \tag{3.17}$$

Putting these solutions in the Lagrangian (3.9) yields the metric

$$ds^2 = d\rho_1^2 + d\rho_3^2 + \frac{\rho_1^2}{k^2} d\tilde{\theta}_1^2 + \frac{\rho_3^2}{(k/(jk-n))^2} d\tilde{\theta}_2^2 \tag{3.18}$$

where the gauge invariant angles are $\tilde{\theta}_1 = (\theta_k - k\theta_1), \tilde{\theta}_2 = \left(\theta_k - \frac{k}{(2k-n)}\right)$, and according to our convention of the last section, the D-term constraint (3.12) necessitates the redefinition of the charge of ϕ_3 as $n \to (n - jk)$, where j is the smallest integer that makes $(jk - n)$ positive. Likewise, when ϕ_1 is made very large, it can be checked that we recover flat space. This describes the decay of the orbifold $\mathbb{C}^2/\mathbb{Z}_{n(k)}$ via turning on a single U(1). The resulting space(s) can of course be singular themselves, in which case it is necessary to turn on a second U(1) and follow its flow, and so on. The details of this can be found in [9].[5]

3.2 Two parameter GLSMs

We now move over to the description of two parameter GLSMs. The calculations are straightforward but lengthy, and we will simply present the final results. In this case, it is easy to check that varying the bosonic action (3.2), we get the following equations for the gauge fields

$$\sum_i Q_i^a \sum_b Q_i^b V_\mu^b = Q_i^a \sum_i \mathrm{Im}\left(\bar{\phi}_i \partial_\mu \phi_i\right) \tag{3.19}$$

where $a, b = 1, 2$, and the sum over i goes over the chiral fields. Eq. (3.19) gives a set of simultaneous equations and can be readily solved, and it is seen that in this case, integrating out V_μ^a, i.e solving for V_μ^a from (3.19), and substituting in the Lagrangian of (3.2) yields, after a somewhat lengthy computation

$$L = L_1 + L_2 \tag{3.20}$$

[5]Note that our GLSM analysis of $\mathbb{C}^2/\mathbb{Z}_{n(k)}$ essentially describes the condensation of the tachyon of the first twisted sector of the closed string CFT. In order to study the condensation of the tachyon in the jth twisted sector, we need to modify the GLSM to have the charges $Q_i = (j, jk, -n)$ and the sigma model metrics can be calculated as usual.

where

$$L_1 = \sum_i (\partial_\mu \rho_i)^2 \tag{3.21}$$

$$L_2 = \frac{\sum_{[i,j,k]} \left[\rho_i \rho_j \rho_k d\theta_i \left(Q_j^b Q_k^a - Q_k^b Q_j^a\right)\right]^2}{\sum_{i<j} \rho_i^2 \rho_j^2 \left(Q_i^b Q_j^a - Q_j^b Q_i^a\right)^2} \tag{3.22}$$

Where we have written $\phi_i = \rho_i e^{i\theta_i}$,[6] and the symbol $[i,j,k]$ in the summation in the numerator of L_2 denotes *cyclic* combinations of the variables. The equations (3.20), (3.21) and (3.22) are the main result of this subsection.

The details of the calculation are unimportant, but as a check, let us consider the single parameter GLSM with three fields having charges

$$Q_i = (1, n_2, -n_3) \tag{3.23}$$

which, in the limit of the Fayet-Iliopoulos parameter going to infinity, describes the unresolved orbifold $\mathbb{C}^2 \mathbb{Z}_{n_3(n_2)}$. The metric calculated from (3.9) yields

$$ds^2 = d\rho_1^2 + d\rho_2^2 + \frac{\rho_1^2}{n_3^2} d\tilde{\theta}_1^2 + \frac{\rho_2^2}{(n_3/n_2)^2} d\tilde{\theta}_2^2 \tag{3.24}$$

where $\tilde{\theta}_1 = \theta_3 + n_3\theta_1$, $\tilde{\theta}_2 = \theta_3 + \frac{n_3}{n_2}\theta_2$ are the gauge invariant angles, with θ_1 and θ_2 appearing in the original solutions of ϕ_1 and ϕ_2. The same metric can be calculated by using a two parameter model, with a second U(1) corresponding to the jth twisted sector of the CFT being turned on. Using the charges

$$Q_i = \begin{pmatrix} n_1 & n_2 & n_3 & 0 \\ jn_1 & jn_2 & 0 & n_3 \end{pmatrix} \tag{3.25}$$

which indeed describes the completely unresolved orbifold $\mathbb{C}^2/\mathbb{Z}_{n_3(n_2)}$ (as can be seen by simply writing down the two D-term constraints), in (3.21) and (3.22), we obtain the sigma model metric

$$ds^2 = d\rho_1^2 + d\rho_2^2 + \frac{\rho_1^2}{n_3^2} d\tilde{\theta}_1'^2 + \frac{\rho_2^2}{(n_3/n_2)^2} d\tilde{\theta}_2'^2 \tag{3.26}$$

where now θ_1' and θ_2' are given by the expressions

$$\theta_1' = (j\theta_4 + \theta_3) + n_3\theta_1, \quad \theta_2' = (j\theta_4 + \theta_3) + \frac{n_3}{n_2}\theta_2 \tag{3.27}$$

This is seen to match with (3.24) by a trivial redefinition of the angles.

Note that in our calculations leading to the Lagrangians in (3.9) and (3.20), we have used the homogeneous coordinates of the GLSM. It is possible to perform the analysis presented here by using symplectic coordinates, where it turns out that the essential information about the metric is given by the Hessian of the *symplectic potential* [29], by

[6] From now on, we will not explicitly solve the D-term constraint. Rather we write the GLSM fields $\phi_i = \rho_i e^{i\theta_i}$ and write the Lagrangian in terms of ρ_is and θ_is. This will help us to find a general expression in which we can make any field acquire an arbitrarily large vev, so that it is integrated out.

generalising an approach due to Guillemin [30] (see also [31]). However, we wish to assert that our calculation above presents the Lagrangian entirely in terms of the U(1) charges of the GLSM, which is the kernel of the toric data of the orbifold singularity, and is completely general in that aspect. We will therefore use the above analysis in a general manner while dealing with generic multi parameter GLSMs.

The analysis presented above can be used to study non-cyclic singularities of the form $\mathbb{C}^3/\mathbb{Z}_m \times \mathbb{Z}_n$, which cannot be described by a single parameter GLSM. Before we embark on the full details in the next section, let us end this section by presenting a couple of results for the metrics of completely unresolved supersymmetric orbifolds. Our first example is the orbifold $\mathbb{C}^3/\mathbb{Z}_{4(1,1,2)}$. This can be described by the two-parameter GLSM with charges

$$Q_i = \begin{pmatrix} 1 & 1 & 2 & -4 & 0 \\ 2 & 2 & 0 & 0 & -4 \end{pmatrix} \tag{3.28}$$

The metric, in this case, is given by

$$ds^2 = d\rho_1^2 + d\rho_2^2 + d\rho_3^2 + \frac{\rho_1^2}{16}d\tilde{\theta}_1^2 + \frac{\rho_2^2}{16}d\tilde{\theta}_2^2 + \frac{\rho_3^2}{4}d\tilde{\theta}_3^2 \tag{3.29}$$

with

$$\tilde{\theta}_1 = (\theta_4 + 4\theta_1 + 2\theta_5)\,, \quad \tilde{\theta}_2 = (\theta_4 + 4\theta_2 + 2\theta_5)\,, \quad \tilde{\theta}_3 = (\theta_4 + 2\theta_3) \tag{3.30}$$

Our last example in this section is the supersymmetric orbifold $\mathbb{C}^3/\mathbb{Z}_2 \times \mathbb{Z}_2$. This can be described by the $\mathrm{U}(1)^2$ GLSM with five fields, having, in a certain basis, the charges[7]

$$Q_i = \begin{pmatrix} 1 & 1 & 0 & -2 & 0 \\ 0 & 1 & 1 & 0 & -2 \end{pmatrix} \tag{3.31}$$

The corresponding toric data being

$$\mathcal{T} = \begin{pmatrix} 1 & 0 & 0 & 0 & 2 \\ 0 & 1 & 0 & -1 & 1 \\ 0 & 0 & 1 & 2 & -2 \end{pmatrix} \tag{3.32}$$

The sigma model metric, calculated from (3.20) yields

$$ds^2 = d\rho_1^2 + d\rho_2^2 + d\rho_3^2 + \frac{\rho_1^2}{4}d\tilde{\theta}_1^2 + \frac{\rho_2^2}{4}d\tilde{\theta}_2^2 + \frac{\rho_3^2}{4}d\tilde{\theta}_3^2 \tag{3.33}$$

where we have defined

$$\tilde{\theta}_1 = (\theta_4 + 2\theta_1)\,, \quad \tilde{\theta}_2 = (\theta_5 + 2\theta_3)\,, \quad \tilde{\theta}_3 = (\theta_4 + \theta_5 + 2\theta_2) \tag{3.34}$$

3.3 General r parameter GLSMs

We are now ready to write down the result for the Lagrangian for the general r parameter GLSM. Once again, the details are unimportant (as much as they are lengthy), and we

[7] In the notation of the last section, we have turned on two of the three twisted sectors, corresponding to the charges $\left(\frac{1}{2}, \frac{1}{2}, 0\right)$ and $\left(\frac{1}{2}, 0, \frac{1}{2}\right)$

simply present the result here. We find that in the general case, the r parameter GLSM Lagrangian can be written as

$$L = L_1 + L_2 \tag{3.35}$$

where now

$$L_1 = \sum_i (\partial_\mu \rho_i)^2 \tag{3.36}$$

$$L_2 = \frac{\sum_{[j_1,j_2,\ldots,j_{r+1}]} \left[\rho_{j_1}\rho_{j_2}\cdots\rho_{j_{r+1}}\partial_\mu\left(\theta_{j_1}K_{j_2,\ldots,j_r}\right)\right]^2}{\sum_{j_1<j_2<\cdots j_r}\rho_{j_1}^2\rho_{j_2}^2\cdots\rho_{j_r}^2\left[\Delta\left(j_1,j_2,\ldots,j_r\right)\right]^2} \tag{3.37}$$

where i and $j_1, j_2, \ldots, j_{r+1}$ go from $1, 2, \ldots, r+m$ for a $\mathbb{C}^m/\Gamma$ orbifold, $K_{j_2,\ldots,j_r}$ is the j_1th component of the Kernel of the matrix formed by the charges of the j_{r+1} vectors in the numerator of L_2 (and hence depends on $j_2, \ldots j_{r+1}$), and $\Delta\left(j_1, j_2, \ldots, j_r\right)$ is the determinant of the matrix formed by the charge vectors $\rho_{j_1}, \rho_{j_2} \cdots \rho_{j_r}$ under the r U(1)s. Also, as before, the notation $[j_1, j_2, \ldots, j_{r+1}]$ indicates a cyclic combination of the variables.[8]

The advantage of the formula (3.35) above is that it is completely general and we have written it *entirely* in terms of the toric data of the orbifold. One can check that in the special cases of the one and the two parameter GLSMs, eq. (3.35) reduces to the eqs. (3.9) and (3.20) respectively. Eq. (3.35) is the main result of this subsection.

The formulation above can be applied to study the dynamics of D-branes in GLSMs, by using the boundary GLSM approach of [11, 12]. In particular, we may hope that by using the open string world sheet description of the GLSM Lagrangian that we have discussed, we might be able to derive general D-brane boundary conditions, for multi parameter GLSMs, and study branes in various phases of these. We will comment on this toward the end of the paper.

Having written down the GLSM Lagrangian in its most general form entirely in terms of the toric data of the orbifold, we can now use this formalism to gain knowledge about the phases of the GLSMs for any given orbifold. Essentially these phases will be obtained from the Lagrangian by making some fields in the GLSM very large. Since we have the Lagrangian in the most general form, we do not need to explicitly solve the D-term constraints[9] but can work directly with the fields.

As a check on our formula, note that for $r = 2$, it reduces to eq. (3.20). Consider a generic $\mathbb{C}^3/\Gamma$ singularity, with five fields and a $\mathrm{U}(1)^2$ charge matrix being given by

$$Q = \begin{pmatrix} Q_1^1 & Q_2^1 & Q_3^1 & Q_4^1 & Q_5^1 \\ Q_1^1 & Q_2^1 & Q_3^1 & Q_4^1 & Q_5^1 \end{pmatrix} \tag{3.38}$$

[8] Operationally, the process of giving large vevs to certain fields so as to integrate them out is similar to resolving certain points in the toric diagram corresponding to the singularity. It may look like there is a potential ambiguity in the definition of the charges when certain fields are resolved, since the squares of the charges appear in the denominator of (3.37). However, this can be fixed by appealing to the consistency of the toric data at each step. We will have more to say about this toward the end of this section.

[9] Excepting that in eq. (3.36) we need to set the terms corresponding to the "large fields" to zero, since they are implicitly solved by constraints of the form (3.5) and (3.13), and as can be easily seen, drop out of the calculation

where the subscripts on the charges label the fields and the superscripts label the $\mathrm{U}(1)$. Now, in the Lagrangian of eq. (3.20), we take the classical limit $\rho_1 \to \infty$. It is easy to see that in this limit, the Lagrangian simplifies to one of four fields, charged under a single $\mathrm{U}(1)$, with the charges now being

$$Q = \left(Q_1^1 Q_2^2 - Q_1^2 Q_2^1,\ Q_1^1 Q_3^2 - Q_1^2 Q_3^1,\ Q_1^1 Q_4^2 - Q_1^2 Q_4^1,\ Q_1^1 Q_5^2 - Q_1^2 Q_5^1\right) \tag{3.39}$$

These are the relations that define the new D-term constraints in terms of the single $\mathrm{U}(1)$.[10] Operationally, this is equivalent to removing one point from the toric data given by the original charge matrix, and taking the kernel of the new toric data so obtained to get the new charge matrix. For the case of generic $\mathrm{U}(1)^r$ GLSMs, general expressions for the charges of the fields in the reduced Lagrangian get algebraically complicated, and we will not present the results here. We have checked that in general, by taking the classical limits in which certain fields are made very large reduces the Lagrangian of eq. (3.35) to an appropriate Lagrangian with a lower rank gauge group.

Before we end this section, a few comments are in order. First of all, consider the supersymmetric orbifolds of the form $\mathbb{C}^3/\mathbb{Z}_n$, with the $\mathrm{U}(1)$ charge matrix being given by

$$Q = (Q_1, Q_2, Q_3, Q_4) \tag{3.40}$$

with $\sum Q_i = 0$. In the limit when the Fayet-Iliopoulos parameter is very large (so that one of the fields, say with charge Q_4 is always large and positive), we can consider the GLSM Lagrangian with one of the other fields set to zero, so that our Lagrangian now represents an appropriate $\mathbb{C}^2$ singularity. There are three of these, and when we add them up, we reproduce (apart from a trivial factor of 2) the original GLSM Lagrangian in this limit. This is one of the results of [32], termed "champions meet" in that paper.

Secondly, note for the special case of product spaces, the GLSM Lagrangian of eq. (3.35) is separable. Consider, for example, the surface $\mathcal{F}_0 = \mathbb{CP}^1 \times \mathbb{CP}^1$. This is described by the GLSM with four fields, and a $\mathrm{U}(1)^2$ charge matrix

$$Q = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix} \tag{3.41}$$

The Lagrangian corresponding to this space can be seen to be a sum of two $\mathbb{CP}^1$ Lagrangians. Note that this is an example of a space that is not an orbifold. It will be interesting to study localised tachyon condensation on these toric non-orbifold spaces, presumably by realising these in terms of branes on the resolution of higher dimensional product orbifolds, in lines with [33], and suitably introducing tachyons in the spectrum of the latter theories. We believe that for such non-orbifold spaces, the Lagrangian in equations (3.35) - (3.37), which has been written entirely in terms of toric data, can be effectively used to study the decays of tachyonic instabilities in the same.

[10] Note that we have to ignore any common numerical factor that appears in the expression for the charges in (3.39). In our Lagrangian of eq. (3.35), these numerical factors will cancel out automatically in (3.37) and in (3.36), we will assume that the fields that have been set very large do not contribute because of the implicit nature of the solution for the D-term constraints

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Also, note that there might be a potential ambiguity in the sign of the redefined charges, after giving large vevs to certain fields. This is due to the fact that the square of the charge is what appears in the denominator of eq. (3.37) or eq. (3.22). This is not a problem, if we note that by resolving fields in succession, we finally reach a single parameter GLSM, which effectively describes the same singularity. Operationally, the sign of the charge of the remaining fields is obtained by first writing the fields that take large vevs (in decreasing order, as in (3.35)), and then writing the remaining fields. The determinant in the denominator of (3.35) changes sign appropriately, with the absolute value of the charges remaining the same.

Having discussed the generic r parameter GLSMs, We are now ready to use the results of this section to analyse the phases of generic multiparameter GLSMs.

4. Phases of generic multiparameter GLSMs

In this section, we analyse the phases of generic multiparameter GLSMs, using the formalism of the last section. We will not attempt to deal with GLSMs with generic charges, since this is algebraically cumbersome beyond the two parameter case. Also, we leave a general treatment of these phases in lines with [10] for generic orbifolds of $\mathbb{C}^3$, of the form $\mathbb{C}^3/\mathbb{Z}_m \times \mathbb{Z}_n$ for the future. We will rather try to apply the results of the last section to some specific examples. Consider, for example, the orbifold $\mathbb{C}^3/\mathbb{Z}_{13(1,2,5)}$ [10].[11] Closed string tachyon condensation can be studied in this Type 0 example. There are three twisted sector tachyons in the closed string spectrum, and the $\mathrm{U}(1)^3$ charge matrix is given by

$$Q = \begin{pmatrix} 1 & 2 & 5 & -13 & 0 & 0 \\ 8 & 3 & 1 & 0 & -13 & 0 \\ 3 & 6 & 2 & 0 & 0 & -13 \end{pmatrix} \tag{4.1}$$

This GLSM describes the orbifold with three tachyons, and in order to study the phases of this theory, we can write the GLSM Lagrangian for this charge configuration by specialising to the case $r = 3$ in (3.35). For ease of comparison, we will begin, however, by turning on two of the tachyons, and the GLSM charge matrix for five fields with these two tachyons represented by the last two columns of the charge matrix

$$Q = \begin{pmatrix} 1 & 2 & 5 & -13 & 0 \\ 8 & 3 & 1 & 0 & -13 \end{pmatrix} \tag{4.2}$$

The GLSM Lagrangian can be read off directly from eq. (3.20) or by specialising to the case of $r = 2$ in eq. (3.35). The phase boundaries (in the space of the Fayet-Iliopoulos parameters) can be read off from the columns of the charge matrix (4.2) and the sigma model metrics for the phases of the model can be calculated by considering making two fields large in eq. (3.20). We will simply present the results for the sigma model metrics that can be read off from the Lagrangian in the appropriate limits. In the phases where

[11] This orbifold has been studied in details in [10]. We will, however focus on this example in order for ease of comparison with the methodology of that paper.

the non-tachyonic fields are large, the classical sigma model metrics are

$$
\begin{aligned}
ds_{12}^2 &= d\rho_3^2 + d\rho_4^2 + d\rho_5^2 + \rho_3^2 d\left(\theta_1 - 3\theta_2 + \theta_3\right)^2 + \\
&\quad \rho_4^2 d\left(-3\theta_1 + 8\theta_2 + \theta_4\right)^2 + \rho_3^2 d\left(2\theta_1 - \theta_2 + \theta_5\right)^2 + \\
ds_{13}^2 &= d\rho_2^2 + d\rho_4^2 + d\rho_5^2 + \frac{\rho_2^2}{9} d\left(\theta_1 + \theta_3 - 3\theta_2\right)^2 + \\
&\quad \frac{\rho_4^2}{9} d\left(\theta_1 + \theta_3 - 3\theta_4\right)^2 + \frac{\rho_5^2}{9} d\left(\theta_1 + \theta_3 - 3\theta_5\right)^2 \\
ds_{23}^2 &= d\rho_1^2 + d\rho_4^2 + d\rho_5^2 + \rho_1^2 d\left(-3\theta_2 + \theta_3 + \theta_1\right)^2 + \\
&\quad \rho_4^2 d\left(-\theta_2 + 3\theta_3 + \theta_4\right)^2 + \rho_1^2 d\left(5\theta_2 - 2\theta_3 + \theta_5\right)^2 +
\end{aligned}
\tag{4.3}
$$

where each of the five fields $\phi_i,\ i = 1, \ldots, 5$ have been written as $\phi_i = \rho_i e^{i\theta_i}$ and the subscripts of ds^2 label the fields that have been given large vevs, and in the space of the Fayet-Iliopoulos parameters denote the region in which the metric is valid. It can be seen from above that whereas the first and third regions. represent flat space, the second is the unresolved phase of the supersymmetric GLSM with charges $Q = (1, 1, 1, -3)$, by an appropriate redefinition of the gauge invariant angles. In the same spirit, we can analyse the sigma model metrics for the other regions of moduli space. We summarise the results below

$$
\begin{aligned}
& ds_{14}^2 \sim (1, 1, 3, -8)\,, \quad ds_{15}^2 \sim \text{flat}, \quad ds_{23}^2 \sim \text{flat}, \\
& ds_{24}^2 \sim (-1, 1, -1, -3)\,, \quad ds_{25}^2 \sim (1, 1, 1, -2)\,, \\
& ds_{34}^2 \sim \text{flat}, \quad ds_{35}^2 \sim (1, 1, 3, -5)\,, \quad ds_{45}^2 \sim (1, 2, 5, -13)
\end{aligned}
\tag{4.4}
$$

In the above equation, we have written the metrics implicitly, e.g. when the fields $\phi_1, \phi_4 \gg 0$, the resulting metric is that of a GLSM with charges $(1, 1, 3, -8)$ in the infrared limit (i.e the unresolved phase of the latter). Eq. (4.4) above gives us the behaviour of the classical metrics in different regions of moduli space. The distinct *phases* of the theory can be read off straightforwardly, by listing the massless fields that remain after certain fields have been given large vevs. E.g., noting that in the two dimensional space of the Fayet-Iliopoulos parameters, the phase boundaries are vectors denoted by their charges in (4.2) [10] and e.g. the overlap region where ds_{15}^2, ds_{25}^2, ds_{35}^2 are all valid gives the region in moduli space where the tachyon field ϕ_4 is massless, we see that this region is the phase denoting partial resolution by the tachyon field ϕ_4. A similar logic for the other metrics gives the phase diagram for this model. Importantly, we can read off the sigma model metrics not only in the classical regions of all the phases, but other generic points also, i.e whenever some fields in the GLSM acquire large vevs.

A similar analysis can be carried out for this model with all three tachyons in eq. (4.1) turned on. This is straightforward and we will not reproduce the results here. Let us however point out that there is an important difference in calculating the phases of the GLSM with more than two $\mathrm{U}(1)$ gauge fields. Namely, there might be extra columns in the charge matrix corresponding to the intersection of convex hulls obtained using triplets of the original field vectors [10]. These can be obtained from our analysis by studying the Lagrangian (for the three parameter case) with two of the six fields acquiring large vevs at a time, and then computing the D-term equations for the reduced Lagrangian (obtained by suitable linear combinations of the D-term equations of the original GLSM). In this particular example, one can check that out of the possible 15 combinations where two of the six original fields in eq. (4.1) take large vevs, eight yield independent Lagrangians, and writing down the D-term equations for these, we get precisely the points listed in [10]. We have checked that this holds for generic examples. The entire information of the phases are contained in the GLSM Lagrangian, and can be obtained using our methods.

Finally, we will briefly focus on Abelian non-cyclic orbifolds of $\mathbb{C}^3$, of the form $\mathbb{C}^3/\mathbb{Z}_m \times \mathbb{Z}_n$. The analysis of the previous section can be applied here, once we write down the charge matrix of the GLSM appropriate to the singularity. We will start here with the simplest example of such a singularity that has tachyons in the spectrum: the orbifold $\mathbb{C}^3/\mathbb{Z}_2 \times \mathbb{Z}_5$ where the orbifolding action is given by

$$\begin{aligned} g_1 &: \left(Z^1, Z^2, Z^3\right) \rightarrow \left(-Z^1, -Z^2, Z^3\right) \\ g_2 &: \left(Z^1, Z^2, Z^3\right) \rightarrow \left(Z^1, \omega Z^2, \omega^2 Z^3\right) \end{aligned} \tag{4.5}$$

where $\omega = e^{\frac{2\pi i}{5}}$. There are four twisted sectors here, with fractional R-charges $\left(\frac{1}{2}, \frac{1}{2}, 0\right)$, $\left(0, \frac{1}{5}, \frac{2}{5}\right)$, $\left(0, \frac{3}{5}, \frac{1}{5}\right)$, $\left(\frac{1}{2}, \frac{1}{10}, \frac{1}{5}\right)$, of which the first is marginal and the others are relevant. We will illustrate our method of the previous section by focusing on a single tachyon, i.e we consider turning on the twisted sectors corresponding to the R-charges $\left(\frac{1}{2}, \frac{1}{2}, 0\right)$ and $\left(0, \frac{1}{5}, \frac{2}{5}\right)$. In this case, the GLSM charge matrix calculated from the toric data is

$$Q = \begin{pmatrix} 1 & 1 & 0 & -2 & 0 \\ 0 & 1 & 2 & 0 & -5 \end{pmatrix} \tag{4.6}$$

where the first D-term is supersymmetric. Using the Lagrangian in eq. (3.20), we can compute the sigma model metrics (and the phases) of this theory. For example, the unresolved orbifold has the metric[12]

$$ds_{45}^2 = \sum_{i=1}^{3} d\rho_i^2 + \frac{\rho_1^2}{(2)^2} d\tilde{\theta_1}^2 + \frac{\rho_1^2}{(10)^2} d\tilde{\theta_2}^2 + \frac{\rho_3^2}{(2)^2} d\tilde{\theta_3}^2 \tag{4.7}$$

where the ρ_i are the real parts of ϕ_i and the gauge invariant angles are now $\tilde{\theta_1} = 2\theta_1 + \theta_4$, $\tilde{\theta_2} = 10\theta_2 + 5\theta_4 + 2\theta_5$, $\tilde{\theta_3} = 5\theta_3 + 2\theta_5$. Most of the phases of this model are flat. In the region of moduli space where ϕ_1 and ϕ_5 acquire large vevs (this is a distinct phase of the theory), we obtain the metric

$$ds_{15}^2 = d\rho_2^2 + d\rho_3^2 + d\rho_4^2 + \frac{\rho_2^2}{(5)^2} d\tilde{\theta_2}^2 + \frac{\rho_3^2}{(5)^2} d\tilde{\theta_3}^2 + \rho_4^2 d\tilde{\theta_4}^2 \tag{4.8}$$

where now $\tilde{\theta_2} = 5(\theta_1 - \theta_2) + \theta_5$, $\tilde{\theta_3} = 5\theta_3 + 2\theta_5$, $\tilde{\theta_4} = 2\theta_1 + 2\theta_4$, which is recognised to be the metric for the space $\mathbb{C}^2/\mathbb{Z}_{5(2)} \times \mathbb{C}$. Similarly in the region of moduli space where ϕ_1 and ϕ_3 acquire large vevs, the metric is seen to be that of $\mathbb{C}^2/\mathbb{Z}_{2(1)} \times \mathbb{C}$. This is the same

[12] We use the same notation as the previous section

metric seen in the region of moduli space where ϕ_3 and ϕ_4 acquire large vevs. The latter is of course a distinct phase of the theory as can be seen by plotting the charge vectors.

We conclude by briefly commenting on the orbifold $\mathbb{C}^3/\mathbb{Z}_5 \times \mathbb{Z}_5$. We consider the two-tachyon system with the relevant twisted sectors with R-charges $\left(\frac{1}{5}, \frac{2}{5}, 0\right)$ and $\left(0, \frac{1}{5}, \frac{2}{5}, 0\right)$ being turned on. The GLSM charge matrix in this case can be seen to be

$$Q = \begin{pmatrix} 1 & 2 & 0 & -5 & 0 \\ 0 & 1 & 2 & 0 & -5 \end{pmatrix} \tag{4.9}$$

It is straightforward to compute the five distinct phases in this example. In the unresolved phase, the classical sigma model metric is given by

$$ds_{45}^2 = \sum_{i=1}^{3} d\rho_i^2 + \frac{\rho_1^2}{(5)^2} d\tilde{\theta_1}^2 + \frac{\rho_2^2}{(5)^2} d\tilde{\theta_2}^2 + \frac{\rho_3^2}{(5)^2} d\tilde{\theta_3}^2 \tag{4.10}$$

Where $\tilde{\theta_1} = 5\theta_1 + \theta_4$, $\tilde{\theta_1} = 5\theta_2 + 2\theta_4 - \theta_5$, $\tilde{\theta_3} = 5\theta_3 + 2\theta_5$. When, for example, the fields ϕ_1 and ϕ_5 take very large vevs, it is easy to see that the metric for this region is that of $\mathbb{C}^2/\mathbb{Z}_{5(2)} \times \mathbb{C}$.

Of course in order to understand the possibilities of flop transitions etc. in these generic orbifolds, one needs to further analyse this system including all the tachyons. We will leave such an analysis to a future publication.

5. Conclusions

In this paper, we have carried out a GLSM analysis of generic orbifolds of $\mathbb{C}^2$ and $\mathbb{C}^3$. We have constructed the most general form of the GLSM Lagrangian, entirely in terms of the toric data for the singularity which the model describes, from which the phases for these models can be read off. Our methods, which are applicable to GLSMs with arbitrary charge matrices generalise the results of [10, 27]. These can, in particular, be used to study the phases of generic GLSMs in lines with [27] which may not have an SCFT description.

As we have pointed out in section 3, it would be interesting to use the expression for the general multi parameter GLSM that we have derived in (3.35), to study D-brane dynamics in these models, in the lines of [11, 12]. In these papers, single parameter GLSMs were used to study supersymmetric Calabi-Yau target spaces. In particular, in [12], D-branes in the non-linear sigma model limit of the open string GLSMs (i.e GLSMs with boundaries) were studied for the single parameter case, and was already seen to give rise to some interesting physics. The same analysis can be carried out using our results for general GLSMs with boundaries, with arbitrary number of gauge groups and arbitrary charges. The behaviour of D-branes in various phases of these models can be studied using the results here, and is expected to yield a rich structure. Further, we can also study D-branes in non-supersymmetric backgrounds using these. In particular, the roles of fermions in these theories (with consistent Type II GSO projection) need to be understood. We leave such a detailed study for the future.

Finally, it would be interesting to understand the full phase structure of generic orbifolds of the form $\mathbb{C}^3/\mathbb{Z}_m \times \mathbb{Z}_n$, and possibilities of flop transitions therein, from the analysis that we have presented here. This can be carried out in a straightforward manner from the analysis presented here.

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